The Swift-Hohenberg equation requires non-local modifications to model spatial pattern evolution of physical problems

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Abstract

I argue that "good" mathematical models of spatio-temporal dynamics in two-dimensions require non-local operators in the nonlinear terms. Consequently, the often used Swift-Hohenberg equation requires modification as it is purely local. My aim here is to provoke more critical examination of the rationale for using the Swift-Hohenberg equations as a reliable model of the spatial pattern evolution in specific physical systems.

1 Introduction

Consider the spatio-temporal dynamics of systems with a very large horizontal extent, when compared to their height. For two definite examples, I will refer to Rayleigh-Benard convection,

$$\begin{aligned} \tfrac{1}{Pr}\left(\tfrac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}\right) &= -\nabla p' + Ra\,\theta\mathbf{e}_z+\nabla^2\mathbf{u}\,, \\ \tfrac{\partial\theta}{\partial t}+\mathbf{u}\cdot\nabla\theta &= w+\nabla^2\theta\,, \end{aligned} \tag{1}$$

where Pr and Ra are the Prandtl and Rayleigh numbers respectively, and also refer to a toy set of partial differential equations,

$$\begin{aligned} \tfrac{\partial a}{\partial t} &= ra-\left(1+\nabla^2\right)^2 a - ab\,, \\ \tfrac{\partial b}{\partial t} &= -\left(\alpha_0-\alpha_1\nabla^2+\alpha_2\nabla^4\right)b+a^2\,. \end{aligned} \tag{2}$$

For values of the parameters near some critical value, a continuum of modes, an annulus in Fourier space $|\mathbf{k}| \approx \text{const}$, become linearly unstable. Through physical nonlinearities these modes not only saturate, but also interact. The

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interaction of the continuum of critical modes is the main feature of interest as it determines the spatial patterns in the horizontal. In order to explore pattern evolution, many researchers [2, 4, 5, 7, 9, 10, 13, for example] have invoked the Swift-Hohenberg [16] equation,

$$\frac{\partial A}{\partial t} = \mu A - \left(k_0^2 + \nabla^2\right)^2 A - \gamma A^3 \,, \tag{3}$$

as a useful system to study. Whereas it is indeed instructive to examine the Swift-Hohenberg equation, I contend that the Swift-Hohenberg equation is deficient as an accurate and reliable model of most specific physical planform evolution problems.

Instead, I propose that it is generally necessary to incorporate non-local nonlinearities into a low-dimensional model of planform evolution. Specifically, I recommend an equation of the form

$$\frac{\partial A}{\partial t} = \mu A - \left(k_0^2 + \nabla^2\right)^2 A - A\mathcal{G} \star A^2 \,, \tag{4}$$

where $\mathcal{G}\star$ is some radially symmetric convolution. My argument to support this recommendation has three facets. Firstly, symmetry considerations permit non-local operators in a model (§2). Secondly, non-local operators are naturally generated in systematic methods of modelling nonlinear pattern evolution [14] (§3). Indeed, both Swift & Hohenberg [16] and Bestehorn *et al* [1] naturally encountered non-local terms in their derivations; but they heuristically argued to replace them by local nonlinearities. Thirdly, the range of Fourier harmonics generated by the nonlinearities is fundamentally different in two-dimensions than in one-dimension (§4). In the two-dimensions of a planform evolution problem nonlinearities generate a continuous disc in Fourier space of harmonics, whereasin one-dimension the generated harmonics congregate in discrete lumps. This difference requires a more sophisticated treatment of the two-dimensional planform evolution problem, one that necessarily leads to non-local nonlinearities.

2 Symmetry arguments permit non-local effects

The rationale behind the use of the Swift-Hohenberg equation (3) is firstly that for parameter values near critical, $\mu \approx 0$, the spectrum of the linear terms, $\lambda = \mu - \left(k_0^2 - |\mathbf{k}|^2\right)^2$, matches, near the critical wavenumber k_0, the spectrum of the physical problem under study. Secondly, that the cubic nonlinearity is typical for the symmetry of many systems and is needed to stabilise finite amplitude dynamics. The symmetry invoked is that of sign: $A \to -A$ leaves the Swift-Hohenberg equation (3) unchanged; as does $a \to -a$ in the toy problem (2), and $(\theta, w, z) \to (-\theta, -w, -z)$ in Rayleigh-Benard convection (1). This symmetry is only consistent with odd functions. However, this symmetry applies to the field

as a whole, not to every part of it individually. Thus a wide variety of cubic functions are potentially permissible: for example, non-local functions such as $A(\mathbf{x})^2 A(\mathbf{x}+\mathbf{e}_x)$ or

$$\int K(\mathbf{x},\mathbf{x}',\mathbf{x}'',\mathbf{x}''')A(\mathbf{x}')A(\mathbf{x}'')A(\mathbf{x}''')\,d\mathbf{x}'d\mathbf{x}''d\mathbf{x}''' ,$$

where $\mathbf{x}=(x,y)$ etc.

However, translational and rotational symmetry in space requires that any non-local effects must be expressible as radially symmetric convolutions. The local nature of typical models of physical dynamics implies that harmonics are forced by locally expressible functions. But the feedback from the harmonic to the critical modes, essential for stabilisation of finite amplitude dynamics, need not be local. Indeed a number of systematic studies have shown that memory, either temporal or upstream, are needed in the low-dimensional modelling of forced dynamics [3] or shear dispersion in varying channels [15, 12]. In a planform evolution problem such a "memory" of horizontal structure, occurring through fluid convection for example, would manifest itself as the non-local convolution in a cubic nonlinearity, such as the $A\mathcal{G}\star A^2$ term in (4) where $\mathcal{G}$ is some radially symmetric kernel.

3 Non-local nonlinearities are natural

The evolution of the Swift-Hohenberg equation contains a thin annulus of critical modes, $|\mathbf{k}|\approx k_0$, near onset. It also contains a wide variety of non-critical, exponentially damped modes, $|\mathbf{k}|\not\approx k_0$. Thus the interesting long-term evolution of the critical modes is embedded within the equation along with the evolution of many uninteresting modes. It is for just such a scenario that I developed the concept of an embedded centre manifold [14]. There I show that adiabatic iteration, namely the repeated application of adiabatic elimination [11, 18, 17], effectively embeds the critical modes of a slow manifold into the dynamics of a higher-dimensional system. In a pattern evolution problem, the state space of the higher-dimensional system consists of all the modes in the two-dimensional plan, whereas the slow manifold is composed of just the annular neighbourhood of the critical modes.

For the toy problem (2) with $\alpha_0=1-r$, $\alpha_1=2$ and $\alpha_2=1$, adiabatic iteration leads to (4) as a first non-trivial approximation where

$$\left[(1-\nabla^2)^2-r\right]\mathcal{G}=\delta(\mathbf{x})\,. \tag{5}$$

For example, if $r=0$ then $\mathcal{G}=\frac{1}{4\pi^2}K_0(\mathbf{x})\star K_0(\mathbf{x})$ in two-dimensions and $\mathcal{G}=\frac{1}{4}(1+|x|)\exp(-|x|)$ in one-dimension.

Now the left-hand side of (5) comes directly from the spectrum of the exponentially decaying branches of the linearised problem. The only way to avoid a

Figure 1: schematic diagram in wave-number space of the leading order nonlinear interactions between modes, $\mathbf{k}_1$ and $\mathbf{k}_2$, on the critical circle $|\mathbf{k}| = k_0$ through the nonlinear generation of the harmonic at point B.

Figure 2: schematic typical amplitude spectrum of two-dimensional pattern evolution where the critical modes with $|\mathbf{k}| = k_0$ have typical amplitude ϵ.

non-local operator is if the solution of the analogue of (5), in the given physical system, is a delta function, $\mathcal{G} \propto \delta(\mathbf{x})$. Given the typically elliptic nature of dissipative and spatially symmetric operators, this can only happen if the operator on the left-hand side of the analogue to (5) is constant as a function of wave-number $|\mathbf{k}|$ (or more generally, constant on each branch). Generally this will not occur; for example, in Rayleigh-Benard convection (1) with stress-free boundaries and $Pr = 1$ the spectrum of the mth branch is

$$\lambda = -m^2 - |\mathbf{k}|^2 \pm \sqrt{Ra}\frac{|\mathbf{k}|}{\sqrt{|\mathbf{k}|^2 + m^2}}\,. \tag{6}$$

Thus it is generic that systematic modelling naturally leads to non-local nonlinearities such as shown in (4).

4 Forced harmonics in Fourier space

Here I argue that the richness of non-local nonlinearities are generally necessary for an accurate model of a physical problem. The argument rests on the basic nature of the power spectrum in the two-dimensional problem. As shown in Figure 1, two critical modes with wavenumbers $\mathbf{k}_1$ and $\mathbf{k}_2$ on the critical circle $|\mathbf{k}| = k_0$ interact through quadratic nonlinearities to generate a harmonic at B, wavenumber $\mathbf{k}_1 + \mathbf{k}_2$. In the toy problem (2) the relevant quadratic nonlinearity is the a^2 term in the b equation; in convection it is the advection terms, $\mathbf{u} \cdot \nabla \mathbf{u}$ and $\mathbf{u} \cdot \nabla\theta$, on the left-hand sides of (1). Then such a forced harmonic interacts with the $-\mathbf{k}_1$ critical mode, via the ab term in (2) or the advection terms in (1) for example, to generate a forcing of the $\mathbf{k}_2$ component in the critical modes. By varying $\mathbf{k}_1$ and $\mathbf{k}_2$ independently around the critical circle, *every* wavenumber within the disc $|\mathbf{k}| < 2k_0$ is independently forced.[1]

Thus the amplitude spectrum of typical planform evolution evolution is as shown in Figure 2. Observe that this amplitude spectrum is qualitatively different from that in one-dimension. In one-dimension the amplitude spectrum consists of discrete lumps at integer multiples, nk_0, of the critical wavenumber. For example, this property of the one-dimensional problem is crucial in

[1] It is symmetry, for example $a \to -a$, which ensures that two critical modes together do not directly force another critical mode.

proofs that a Ginzburg-Landau equation is relevant to one-dimensional pattern evolution[6]. However, in two-dimensional problems the amplitude spectrum is considerably richer. Consequently, in order to model the leading order, physical interactions among all these modes, it is *necessary* to determine the forced harmonics over the entire disk $|\mathbf{k}| < 2k_0$. Because these harmonics naturally decay with a rate which depends upon wavenumber, then in physical space this determination can only be done via a non-local convolution. For example, in the toy problem we solve (5) for use in (4) in order to account for the variations of the decay rate $-\lambda = -r + (1 + |\mathbf{k}|^2)^2$ of the harmonics b over the disc $|\mathbf{k}| < 2k_0$.

The best that the Swift-Hohenberg equation (3) can do is to approximate the functional dependence $\lambda(|\mathbf{k}|)$, as above or in (6), by a constant. It is easy to imagine problems where this approximation would be inadequate. For example, if in the toy problem (2) we choose $\alpha_1 = -4\alpha_2 > 0$, then the decay rate of the harmonic b has a minimum at wavenumbers $|\mathbf{k}| = \sqrt{2}k_0$ and so I expect that a square planform would be preferred because the harmonics involved in the necessary interactions are not damped as strongly as other stable modes. (In convection there can be a minimum in the decay-rate, but for $Pr = 1$, (6), it occurs for $|\mathbf{k}| < k_0$.) The Swift-Hohenberg equation misses such subtleties, and as such it cannot be expected to be a reliable model of any given physical problem. Instead, non-local terms based on the wavenumber dependence of the decay of harmonics need to be used in order to make reliable physical predictions from such a model.

Lastly, I give a further extremely formal argument for (4), one based on the new notion of matching centre manifolds[19] in order to systematically develop models of specific problems. The idea is to develop a model, of lower-dimension than the original dynamical system, whose slow manifold evolution matches that of the original system to some order of analysis. This principle is very like that of using Padé approximants to sum Taylor series. Here I would propose the modified Swift-Hohenberg equation (4) as a model, and then determine the necessary constraints on the non-local convolution, $\mathcal{G}\star$, in order for the slow evolution of the critical modes of (4) to be the same as those of the physical problem under study.

Here the most restricted version of critical modes are those precisely on the circle $|\mathbf{k}| = k_0$. Thus we express the solution field as, for example,

$$A(\mathbf{x}, t) = \int_0^{2\pi} Z_\phi(t) \exp(i\mathbf{k}_0(\phi) \cdot \mathbf{x}) d\phi + \cdots ,$$

for some complex amplitudes Z. Concentrating only upon the nonlinear interactions, the evolution of the slow modes would take the form

$$\frac{\partial Z_\phi}{\partial t} = \cdots + \int_0^{2\pi} \beta(\phi, \psi) Z_\phi Z_\psi Z_{\pi+\psi}\, d\psi$$

Recently, Edwards & Fauve[8] have in essence used a discretised version of of this, their equation (8), in studying pattern selection in the Faraday experiment.

Indeed their interaction diagram, Figure 4, is essentially the same as Figure 1 above. In order for the matching to take place to cubic order, it is necessary that the interaction coefficient, $\beta(\phi, \psi)$, for the physical problem and for the model to be identical. As explained before, this interaction takes place through all the forced harmonics in the disc $|\mathbf{k}| < 2k_0$. In essence, (ϕ, ψ) appearing in the interaction integral above is a straightforward parameterisation of this disc. (For example, rotational symmetry in the problems imply that β is purely a function of the difference $\phi - \psi$.) Thus, the only way for the model (4) to accurately match a specific physical problem, is for the interaction over all the harmonics in the disc to be accurately represented. Thus non-local nonlinearities are essential.

The only freedom this matching principle permits, up to third order in the analysis, is the freedom to vary the Fourier transform of $\mathcal{G}$ for wavenumbers greater than $2k_0$. This cannot change the non-local nature of the convolution. In general, the Swift-Hohenberg equation requires non-local modification.

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